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$$\text{arc } TP' = \frac{\rho}{2r+\rho}\text{arc } PT = \text{arc } ST,$$

$$\text{arc } P'T' = \text{arc } TP'T' - \text{arc } TP' = \frac{\rho}{2r+\rho}\text{arc } SK^* - \frac{\rho}{2r+\rho}\text{arc } PT$$

$$= \frac{\rho}{2r+\rho}\text{arc } TK = \text{arc } T'K'.$$

This proves that as C_M rolls on C_F, $C_{M'}$ rolls on $C_{F'}$, that is, as P traces an epicycloid so does P'. Now the ratio of the radii of $C_{M'}$ and $C_{F'}$ is

$$\frac{\rho r}{2r+\rho} \div \frac{\rho^2}{2r+\rho} = \frac{r}{\rho},$$

which is the ratio of the radii of C_M and C_F. Therefore the epicycloid traced by P is similar to that traced by P'.

AN ELEMENTARY THEORY OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By EDWARD V. HUNTINGTON, Harvard University.

In most textbooks on the calculus, the proofs of the formulas for differentiating the logarithmic and exponential functions are either confessedly incomplete, or are made to depend on a preliminary study of the complicated function, $y = \lim_{x=\infty} (1 + 1/x)^x$. This function represents one of the most difficult of the indeterminate forms, the study of which would seem more properly to come late in the course, instead of at the beginning. Moreover, the usual treatment passes over altogether too lightly the questions connected with the existence and meaning of the function a^x for irrational values of x—questions which the student can hardly be supposed to have solved satisfactorily in his previous course in algebra.

The present paper is an attempt to develop the theory of logarithms and exponents, including existence theorems and rules for differentiation, in a new way, which it is hoped will prove not only rigorous but teachable. The discussion is confined to the case of the real variable, and *no knowledge of algebra beyond positive integral exponents is pre-supposed.*

Definition of the Exponential Curve. Let us begin by supposing that in the process of plotting a variety of different curves, some one hit upon the idea of plotting the family of curves representing the following simple algebraic functions:

$$(A) \qquad (1 + x/2)^2,\ (1 + x/4)^4,\ (1 + x/16)^{16},\ \cdots (1 + x/m)^m,\ \cdots$$

$$(B) \qquad \frac{1}{(1 - x/2)^2},\ \frac{1}{(1 - x/4)^4},\ \frac{1}{(1 - x/16)^{16}},\ \cdots \frac{1}{(1 - x/m)^m},\ \cdots,$$

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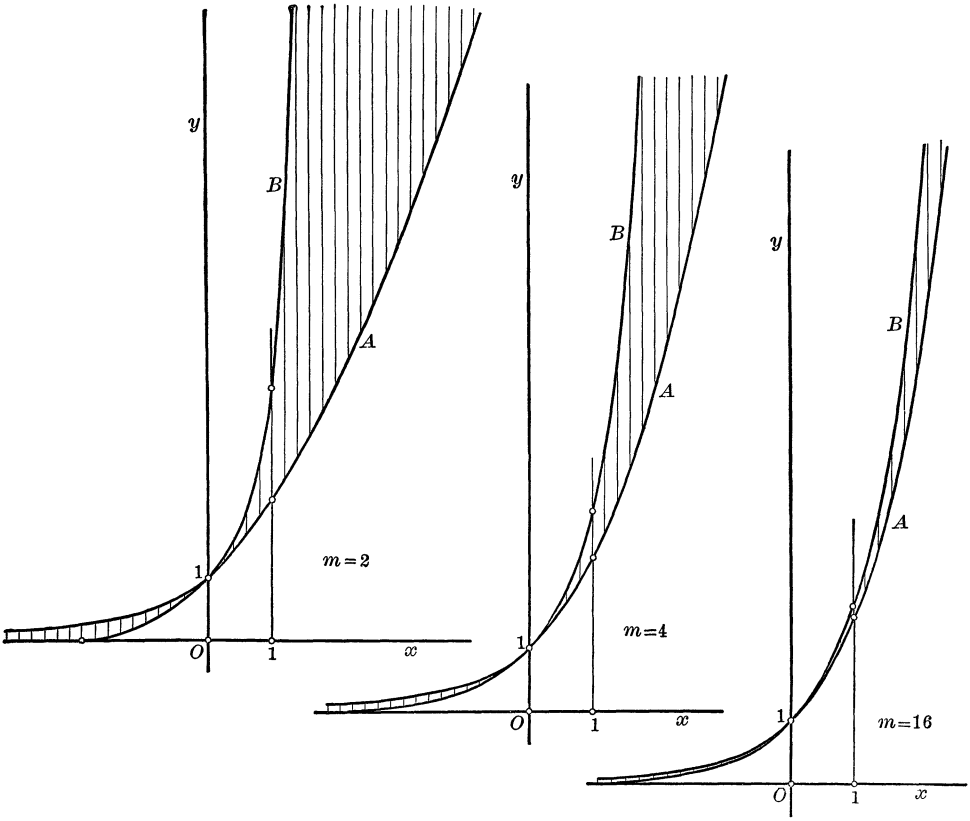
* Since SK is equal to a semicircumference of C_M.

where m runs through the sequence of numbers 2, 4, 16, 256, $\cdots$, each of which is the square of the preceding. It is to be understood, however, that only those values of x are considered which lie between $-m$ and $+m$. Under this restriction, the quantities in parentheses will always be positive. A few of these curves are shown in the accompanying figure.



Each of these functions has the value 1 when $x = 0$.

Fixing now our attention upon any given value of x (not zero), we proceed to prove the following facts, in corroboration of what is apparent from the figure.

(a) For any given value of x the "A" curves in the figure (beginning with the first value of m which is $> |x|$) form an ascending sequence, while

(b) the "B" curves form a descending sequence. Moreover,

(c) each "A" value is less than the corresponding "B" value, and

(d) the difference between corresponding values of the "B" and "A" curves approaches zero when m increases.

Hence, for a given value of x, the "A" sequence and the "B" sequence converge toward a common limit; and this limit is what we shall take as the definition of the *exponential of* x, denoted by $\exp x$. The curve for $y = \exp x$ forms a boundary line between the "A" curves below it and the "B" curves above it.

Proofs of Statements (*a*)–(*d*). In order to prove the foregoing statements, (*a*)–(*d*), we first establish the following inequality:

$$(1 + d)^m > 1 + md, \tag{1}$$

where m is a positive integer and d is positive or lies between -1 and 0. The proof consists of two parts.

First, when d is positive, the inequality is obvious from the binomial theorem for positive integral exponents.

Secondly, when d lies between -1 and 0, let $b = 1 + d$ (so that $0 < b < 1$) and note that

$$1 - b^m = (1 + b + b^2 + \cdots + b^{m-1})(1 - b).$$

Hence, since $0 < b < 1$,

$$1 - b^m < (1 + 1 + 1 + \cdots + 1)(1 - b).$$

Therefore, $b^m > 1 - m(1 - b)$, or $(1 + d)^m > 1 + md$.

Having established this inequality, we now prove (*a*)–(*d*) as follows:

(*a*) By (1), $(1 + x/m^2)^m > 1 + x/m$ since $|x/m| < 1$ and therefore $|x/m^2| < 1$. Hence, $(1 + x/m^2)^{m^2} > (1 + x/m)^m$; that is, each of the "$A$" values is greater than the preceding.

(*b*) By (1), $(1 - x/m^2)^m > 1 - x/m$, since $|x/m| < 1$ and therefore $|x/m^2| < 1$. Hence, $(1 - x/m^2)^{m^2} > (1 - x/m)^m$ or $1/(1 - x/m^2)^{m^2} < 1/(1 - x/m)^m$; that is, each of the "$B$" values is less than the preceding.

(*c*) To prove (*c*), suppose that $(1 + x/m)^m$ were greater than $1/(1 - x/m)^m$; then we should have $(1 - x^2/m^2)^m$ greater than 1, which is impossible, since $x^2/m^2 < 1$. Therefore, $(1 + x/m)^m < 1/(1 - x/m)^m$.

(*d*) To prove (*d*), note that the difference in question is

$$\frac{1}{(1 - x/m)^m} - (1 + x/m)^m = \left[\frac{1}{(1 - x/m)^m}\right][1 - (1 - x^2/m^2)^m].$$

By (*b*), as soon as m is large enough to make $|x/m| < 1$ the first factor decreases. But by (1), as soon as m is large enough to make $x^2/m^2 < 1$, we have $(1 - x^2/m^2)^m > 1 - x^2/m$, so that from this point on, the second factor is less than x^2/m. Hence, the second factor approaches zero, while the first factor remains less than a finite quantity; therefore, their product approaches zero.

We have thus established the existence of the function $\exp x$ for every real value of x, positive, negative, or zero. In particular $\exp 0 = 1$, and $\exp 1 = 2.718 \cdots$. This latter value can be computed to any desired degree of approximation by evaluating either $(1 + 1/m)^m$, or $1/(1 - 1/m)^m$, for larger and larger values of m, and is here taken as the definition of e:

$$e = \exp 1 = 2.718 \cdots .$$

Fundamental Properties of the Exponential Function. From this definition of $\exp x$ we now proceed to deduce the fundamental theorem:

$$\exp(x+y) = (\exp x)(\exp y).$$

To prove this we show that the ratio $(\exp x)(\exp y)/\exp(x+y) = 1$, as follows:

First, using the "A" approximations, $(\exp x)(\exp y)/\exp(x+y)$ is the limit of

$$\frac{[1+x/m]^m[1+y/m]^m}{[1+(x+y)/m]^m}$$

as m runs through its sequence of increasing values. This expression reduces to $\left[1+\frac{1}{m}\cdot\frac{xy}{m+(x+y)}\right]^m$, which, by (1), is $> 1+\frac{xy}{m+(x+y)}$ when m is sufficiently large; hence, the limit in question must be $\geqq 1$.

Secondly, using the "B" approximations, $(\exp x)(\exp y)/\exp(x+y)$ is the limit of

$$\frac{1/[1-x/m]^m\cdot 1/[1-y/m]^m}{1/[1-(x+y)/m]^m}.$$

This expression reduces to $1 \div \left[1+\frac{1}{m}\cdot\frac{xy}{m-(x+y)}\right]^m$, which, by (1), is $< 1 \div \left[1+\frac{xy}{m-(x+y)}\right]$; hence, the limit in question must be $\leqq 1$.

Comparing these two results, we see that the limit must be precisely 1. That is, $(\exp x)(\exp y)/\exp(x+y) = 1$, or $\exp(x+y) = (\exp x)(\exp y)$.

From this fundamental addition theorem, the following theorems are easily derived:

$$\exp 0 = 1, \qquad \exp(-x) = \frac{1}{\exp x}, \qquad \exp(x-y) = \frac{\exp x}{\exp y};$$

also,

$$\exp n = e^n, \qquad \exp\frac{1}{n} = \sqrt[n]{e} = e^{1/n}, \qquad \exp\frac{m}{n} = \sqrt[n]{e^m} = e^{m/n},$$

where m and n are positive integers.

These last results show that whenever x is a rational number, $\exp x$ is precisely equal to what would be called in algebra e^x, that is, the xth power of $2.718\cdots$. On account of our familiarity with the exponential notation, it is usually convenient to write e^x instead of $\exp x$, even when x is not a rational number; we may therefore use the equation,

$$e^x = \exp x$$

as the definition of e^x in the general case.

In this notation, the fundamental theorem assumes the familiar form

$$e^{x+y} = e^x e^y.$$

The notation exp x may be retained, however, in cases where x is a long expression which cannot conveniently be printed as an exponent.

It is easily seen from the figure that e^x is > 1 when x is positive; hence, whenever $x > y$ (say $x = y + c$), we have $e^x = e^{y+c} = e^y e^c > e^y$, so that e^x is an increasing function.

Moreover, as x increases positively, e^x increases indefinitely, and as x increases negatively, e^x approaches zero.

Derivative of e^x. To prove the existence of the derivative of the function $y = e^x$ and to find its value at any point we form the difference-quotient in the usual way as follows:

$$\frac{\Delta y}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x} = \frac{e^x e^{\Delta x} - e^x}{\Delta x} = e^x \cdot \left[\frac{e^{\Delta x} - 1}{\Delta x}\right].$$

The first factor does not depend on Δx. To find the limit of the second factor, we proceed as follows: By (*a*) and (*b*), and (1), when Δx is small enough to make $-1 < \Delta x < 0$ or $0 < \Delta x < 1$, we have

$$1 + \Delta x < \left(1 + \frac{\Delta x}{m}\right)^m < e^{\Delta x} < 1 \div \left(1 - \frac{\Delta x}{m}\right)^m < \frac{1}{1 - \Delta x},$$

and hence after subtracting 1 and dividing through by Δx;

$$1 < (e^{\Delta x} - 1)/\Delta x < 1/(1 - \Delta x), \text{ if } \Delta x \text{ is positive,}$$

or

$$1 > (e^{\Delta x} - 1)/\Delta x > 1/(1 - \Delta x), \text{ if } \Delta x \text{ is negative.}$$

The left-hand member of this inequality is constantly 1, and the right-hand member approaches 1 as a limit, as Δx approaches zero through any sequence of values whatever; hence the limit of the middle member must also be 1; that is,

$$\lim_{\Delta x \doteq 0} \left[\frac{e^{\Delta x} - 1}{\Delta x}\right] = 1.$$

Therefore the limit of $\Delta y/\Delta x$ is e^x; that is,

$$\frac{dy}{dx} = e^x, \qquad \text{or} \qquad d(e^x) = e^x dx.$$

The formula for differentiation is thus proved at one stroke for all real values of x, positive, negative, or zero.

Definition of Natural Logarithms. Having thus established the well-known shape of the exponential curve, and proved the existence of a tangent at every point, we can at once infer that for every positive quantity N there is a real quantity z such that $e^z = N$. This quantity is then, by definition, the natural logarithm of N:

$$z = \log_e N.$$

The familiar properties of the logarithm follow immediately from the definition, in the usual way.

Derivative of $\log_e x$. The curve for $y = \log_e x$ is the same as the curve for $y = e^x$, reflected in the 45° line of the first quadrant. Hence, the slope of the curve $y = \log_e x$ at any point $x = a$ is the reciprocal of the slope of the curve $y = e^x$ at the corresponding point, $y = a$. From this it follows immediately that

$$\left[\frac{d \log_e x}{dx}\right]_{x=a} = \frac{1}{a}.$$

Or, we may write $y = \log_e x$ in the form $e^y = x$, whence $e^y dy = dx$, or $dy = (1/e^y)dx$, or $dy = (1/x)dx$.

Definition of a^x (a positive). Finally, we define a^x (where a is positive) by means of the general equation

$$\log_e (a^x) = x \log_e a, \qquad \text{or} \qquad a^x = e^{x \log_e a}.$$

When x is a positive integer (or a positive or negative rational number) this definition of a^x reduces at once to the forms which are familiar from elementary algebra.

By means of this definition, all the usual properties of a^x follow immediately from the corresponding properties of e^x; in particular, the inverse of the function 10^x gives immediately the logarithm to the base 10 with all its properties. In terms of $\log_{10}$, the definition of a^x may be written in the form

$$\log_{10} (a^x) = x \log_{10} a, \qquad \text{or} \qquad a^x = 10^{x \log_{10} a};$$

which is the form most convenient for numerical computation.

Conclusion. The method of presentation here suggested will be found to be very much shorter and simpler than any of the older methods that give the complete results (see, for example, Chrystal's *Algebra,* or Stolz's *Allgemeine Arithmetik*), and practically as short as many of the methods of the current textbooks, which give the results only for the rational case.

ELEMENTARY PROOF OF A THEOREM DUE TO F. MORLEY.

By TOBIAS DANTZIG, Indiana University.

In a paper read before the Columbus meeting of the American Mathematical Society, December 30, 1915, Professor H. S. White mentioned a theorem due to Professor F. Morley, and first given by him in a memoir entitled: "On Reflexive Geometry."[1] The theorem follows:

If a ring of five circles be formed, the center of each upon a fixed circle and each

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[1] *Transactions of the American Mathematical Society,* Vol. 8, 1907, pp. 23–24.